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# Open Mapping Theorem and It's Applications

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***Abstract:*** **The open mapping theorem is one of the basic theorems of functional analysis and has wide applications. in this paper we review some of these applications. In particular we consider the Bounded linear operators and the convergence of Fourier series.**

***Keywords:*** **open mapping theorem, basic theorems, functional analysis.**

## 1. OPEN MAPPING THEOREM

The open mapping theorem asserts that a surjective bounded linear operator from a Banach space to another Banach space must be an open map. This result is uninteresting in the finite dimensional situation, but turns out to be very important for infinite dimensional spaces. From history there were several concrete, relevant results in various areas, Banach had the insight to single out the property as a theorem.

A map $f:(X,d)\longmapsto(Y,\rho)$ between two metric spaces is called an open map if $f(G)$ is open in $Y$ for any open set $G$ in$X$. This should not be confused with continuity of a map, namely, $f$ is continuous if $f^{-1}(E)$ is open in $X$ for any open set $E$ in$Y$ . As an example, let us show that every non-zero linear functional on a normed space X is an open map. Indeed, pick $z_0\in X$ with$\Lambda z_0=1$. Such point always exists when the functional $\Lambda$ is non-zero. For any open set $G$in $X$, we claim that $\Lambda$Gis open. Letting$\Lambda x_0\in\Lambda G$, as $x_0\in G$ and$G$is open, there exists some $R>0$ such that $B_R(x_0)$ is contained in $G$. Then $x_0+rz_0\in B_R(x_0)$ for all $r\in(-R,R)$ and $\Lambda(x_0+\mathrm{r}z_0)=\Lambda x_0+r$ imply that $(\Lambda x_0+R,\Lambda x_0-R)\in\Lambda$G,so$\Lambda$Gis open.

Before stating the theorem, let's state a necessary and sufficient condition for a linear operator to be open.

**Lemma (1.1)*:*** Let $T\in L(X,Y)$ when $X$ and $Y$ are normed spaces. $T$ is an open map if and only if the image of a ball under $T$ contains a ball.

Roughly speaking, a linear operator either has "fat" image or it collapses everywhere.

**Proof:** The necessary is obvious. For sufficiency, suppose there exists $D_{r_0}(Tx_1)\subset TB_{R_0}(x_0)$ for some$x_1\in B_{R_0}(x_0)$. By linearity, $D_{r_0}(Tx_1)=D_{r_0}(0)+Tx_1\subset TB_{R_0}(x_0)$ implies

$$D_{r_0}(0)\subset TB_{R_0}(x_0)-Tx_1$$

$$=TB_{R_0}-(x_0-x_1)$$

$$\subset TB_{R_1}(0),R_1=R_0+\|x_0-x_1\|.$$

Let G be an open set in$X$. We want to show that $TG$ is open. So, for

$Tx_0\in TG,x_0\in G$, as $G$is open, we can find a small $\rho>0$ such that$B_\rho(x_0)\subset G$. From the above inclusion,

$$D_\varepsilon(0)\subset TB_\rho(0),\qquad \varepsilon=\rho\frac{r_0}{R_1},$$

Or

$$D_\varepsilon(Tx_0)\subset TB_\rho(x_0)$$

which shows that the ball $D_\varepsilon(Tx_0)$ is contained in$TG$, so $TG$is open.

**Theorem (1.1)*: (Open Mapping Theorem):*** Any surjective bounded linear operator from a Banach space to another Banach space is an open map.

Unlike the uniform boundedness principle here we require both the domain and target of the linear operator to be complete.

***Proof:*** Step 1: We claim that there exists $r > 0$ such that

$$D_r(0) \subset \overline{TB_1(0)}.$$

For, as $T$ is onto, we have

$$Y = \bigcup_1^\infty TB_j(0).$$

By assumption $Y$ is complete, so we may apply Baire theorem to conclude that $\overline{TB_{J_0}(0)}$ contains a ball for some $j_0$, i.e.,

$$D_\rho(y_0) \subset \overline{TB_{J_0}(0)}.$$

Since $TB_{j_0}(0)$ is dense in$\overline{TB_{J_0}(0)}$, by replacing $D_\rho(y_0)$ by a smaller ball if necessary,

we may assume $y_0 = Tx_0$, for some $x_0 \in B_{j_0}(0)$. Then

$$D_\rho(y_0) \subset \overline{TB_{J_0}(0)} \subset TB_R(x_0), R = j_0 + \|x_0\|,$$

So$D_\rho(0) \subset \overline{TB_R(0)}$,

or

$$D_r(0) \subset \overline{TB_1(0)}, \qquad r = \frac{\rho}{R}.$$

Step 2: $D_r(0) \subset TB_3(0)$.

First, note by scaling,

$$D_{\frac{r}{2^n}}(0) \subset \overline{TB_{\frac{1}{2^n}}(0)}, \quad for\ all\ n \geq 0 \qquad (1.1)$$

Letting $y \in D_r(0)$, we want to find $x^* \in B_3(0), Tx^* = y$. We will do this by constructing an approximating sequence.

For $\varepsilon = \frac{r}{2}$ from (4.1) with$n = 0$, there exists $x_1 \in B_1(0)$ such that

$$\|y - Tx_1\| < \frac{r}{2}.$$

As $y - Tx_1 \in D_{\frac{r}{2}}(0)$, for $\varepsilon = \frac{r}{2^2}$, from (4.1) with $n = 1$, there exists $x_2 \in B_{\frac{1}{2}}(0)$

such that

$$\|y - Tx_1 - Tx_2\| < \frac{r}{2^2}.$$

Keep doing this we get $\{x_n\}$, $x_n \in B_{\frac{1}{2^{n-1}}}(0)$ such that

$$\|y - Tx_1 - Tx_2 - \cdots Tx_n\| < \frac{r}{2^n}.$$

Setting $\{z_n\} = \sum_1^n x_j$, we have

$$\|y - Tz_n\| < \frac{r}{2^n}.$$

Let's verify that $\{z_n\}$ is a Cauchy sequence in $X. \forall n, m, m < n$,

$$\|z_n - z_m\| = \|x_{m+1} + \cdots + x_n\|$$

$$\leq \|x_{m+1}\| + \cdots + \|x_n\|$$

$$< \frac{1}{2^m} + \cdots + \frac{1}{2^n} \leq \frac{1}{2^{m-1}} \to 0$$

as $m \to \infty$. From the completeness of $X$ we may set $z^* = \lim_{n\to\infty} z_n$. Let's check that $z^* \in B_3(0)$and $Tz^* = y$. For,

$$\|z_n\| \leq \sum_1^n \|x_j\| \leq \sum_1^n \frac{1}{2^{j-1}} \leq 2 < 3.$$

So $z^*$belongs to the closure of $B_2(0)$, or, in $B_3(0)$: Next,

$$\|y - Tz^*\| \leq \|y - Tz_n\| + \|Tz_n - Tz^*\|$$

$$\leq \frac{r}{2^n} + \|T\|\|z_n - z^*\| \to 0$$

as$n \to \infty, so\ y = Tz^*$.

We have shown that the image of the ball $B_3(0)$ under $T$ contains the ball $D_r(0)$.

Recall that a linear operator is invertible if it is bounded, bijective and with a bounded inverse. The following theorem shows that the boundedness of the inverse comes as a consequence of boundedness and surjectivity of the operator when working on Banach spaces.

**Theorem (1.2):** Let X, *Y be Banach spaces, and* $T: X \to Y$ *a continuous linear map from X onto . The T is an open map.*

***Proof:*** We shall denote $X_r = S(0, r)$, the open ball in $X$centered at 0 with radius$r$, and $Y_r$ the same in$Y$ . It suffices to prove that for any$X_r, T(X_r)$, contains a$Y_s$. Assuming that this has been proved, let $G$ be a non-void open set in$X$, and let$x + X_a \subset G$. Let $Y_b$ be such that $Y_b \subset T(X_a)$. Then

$$T(G) \supset T(x + X_a) = Tx + T(X_a) \supset Tx + Y_b,$$

which shows that T(G) contains a neighborhood of every one of its points, and hence is open.

Since$X = \bigcup_{n=1}^{\infty} nX_{r/2}$ , *and* $Y = T(X) = \bigcup_{n=1}^{\infty} nX_{r/2}$, by Baire category theorem, one of the sets $\overline{nT(X_{r/2})}$ contains a non-void open set. Since themap $y \to ny$is a homeomorphism in $Y, \overline{T(X_{r/2})}$ contains a non-void openset. Thus,

$$\overline{T(X_r)} \supset \overline{T(X_{r/2}) - T(X_{r/2})} \supset \overline{T(X_{r/2})} - \overline{T(X_{r/2})} \supset V - V.$$

Since a map of the form $y \to a - y$ is a homeomorphism, the set $a - V$ is open. Since the set $V - V = \bigcup_{a\in V}(a - V)$ is the union of open sets, it is open, contains 0, and hence contains a $Y_t$.

Let $\epsilon_0 = r/2$, and let $\epsilon_i > 0$ be such that$\sum_{i=1}^{\infty} \epsilon_i < \epsilon_0$. Then, according to the result stated in the preceding paragraph, there is a sequence with $\{t_i, i = 0,1, \dots\}$with$t_i > 0, t_i \to 0$, and such that

$$\overline{T(X_{\epsilon_i})} \supset Y_{t_i}, i = 0,1, \dots \qquad (1.2)$$

Let $y \in Y_{t_0}$. It will be shown that there is an $x \in X_r$such that$Tx = y$. From

(1), with $i = 0$, it is seen that there is an $x_0 \in X_{\epsilon_0}$ such that $\|y - Tx_0\| < t_1$Since $y - Tx_0 \in Y_{t_1}$ , from (1), with $i = 1$, there is an $x_1 \in X_{\epsilon_1}$with $\|y - Tx_0 - Tx_1\| < t_2$. Continuing in this manner, a sequence $\{x_n\}$ may be

defined for which $x_n \in X_{\epsilon_n}$, and

$$\left\| y - T\left(\sum_{i=0}^{n} x_i\right) \right\| < t_{n+1}, n = 0,1, \dots \qquad (1.3)$$

Let $z_m = x_0 + \cdots + x_m$, so that for $m > n, \|z_m - z_n\| = \|x_{n+1} + \cdots + x_m\| < \epsilon_{n+1} + \cdots + \epsilon_m$. This shows that $\{z_n\}$ is a Cauchy sequence, and that theseries$x_0 + x_1 + \cdots$ converges to a point $x$ with

$$\|x\| = \lim_{n\to\infty} \|z_n\| \leq \lim_{n\to\infty} (\epsilon_0 + \epsilon_1 + \cdots + \epsilon_n) < 2\epsilon_0 = r.$$

Since $T$ is continuous, it is seen from (2) that$y = Tx$. Thus it has been shown that an arbitrary ball $X_r$ about the origin in $X$ maps onto a set $TX_r$which contains a ball $Y_s = Y_{t_0}$ about the origin in.

**Theorem (1.3)*:* A** *continuous linear one-to-one map of one Banach space onto all of another has a continuous linear inverse.*

**Proof:** Let $X; Y$ be Banach spaces and $T$ a continuous linear one-to-one map with$TX = Y$. Since $(T^{-1})^{-1} = T$ maps open sets onto open sets by OpenMapping Theorem, the map $T^{-1}$ is continuous. Let$y_1, y_2 \in Y, x_1, x_2 \in X, Tx_1 = y_1, Tx_2 = y_2$, and$\alpha$a scalar. Then,

$$T(x_1 + x_2) = Tx_1 + Tx_2 = y_1 + y_2, T\alpha x_1 = \alpha Tx_1 = \alpha y_1,$$

so that

$$T^{-1}(y_1 + y_2) = x_1 + x_2 = T^{-1}y_1 + T^{-1}y_2,$$

And $T^{-1}(\alpha y_1) = \alpha x_1 = \alpha T^{-1}y_1$. These equations show that $T^{-1}$ is linear.

***Remark:*** The open mapping theorem (Theorem (1.2)) can also be derived from Theorem (1.3) as follows:

Let$K = \{x: Tx = 0\}$. Then $K$ is closed linear subspace of$X$. Let $\tilde{T}: X/K \to Y$be the map induced by $T$. $\tilde{T}$ is one-to-one and continuous, so by

**Theorem (1.4):** it is an open map. Let $\pi: X \to X/K$ be the natural projection.$\pi$is an open map.

So $T = \tilde{T} \circ \pi$ is open.

**Lemma (1.5):** Let $X, Y$ *be normed linear spaces. If there is a linear homeo- morphism between* $X$*and* $Y$, *then either both spaces are complete or both are incomplete.*

**Proof:** If $T$ is such a homeomorphism from $X$to, then there exist positive constants $C, D$such that

$$D\|x\| \leq \|Tx\| \leq C\|x\|.$$

It follows that a sequence is Cauchy (convergent to x) in X if and only if it images under $T$ is Cauchy (convergent to $Tx$) in $Y$ .

**Example 1:** Let $X$ be $C^{(1)}[a.b]$ with the norm$\|f\| = \|f\|_\infty + \|f'\|_\infty$, and $Y$ be$C^{(1)}[a.b]$ with the norm $\|f\| = \|f\|_\infty$. Let T be the identity map from $X$ to.

Then T is continuous, but $T^{-1}$ is not continuous since $Y$ is not complete.

**Example 2:** Let Y be an infinite dimensional real Banach space and let $B = \{y_i: i \in k\}$be a Hamel basis for $Y$ such that $\|y_i\| = 1$ for all $i$. Let $X$ be the set of functions $f$ from $k$ to $\mathbb{R}$such that $f(i) = 0$ for all but finitely many $i$'s. Equip $X$ with the norm defined by$\|f\| = \sum_i |f(i)|$. Then X is an incomplete normed linear space. Define $T: X \to Y$ by $T(f) = \sum_i f(i)y_i$.

Then $T$ is one-to-one continuous linear. $T^{-1}$ is not continuous since $X$ is incomplete (and $Y$ is complete).

**Definition (1.5):** Let $T$ be a linear map whose domain $D(T)$ is a linear subspace of a Banach space $X$, and whose range lies in a Banach space $Y$ . The graph of $T$ is the set of all points in the product space $X \times Y$ of the form $(x, Tx)$ with $x \in D(T)$. The operator $T$ is said to be closed if its graph is closed in the product space$X \times Y$ . An equivalent statement is as follows: The operator $T$ is closed if $x_n \in D(T), x_n \to x, Tx_n \to y$ imply that $x \in D(T)$ and $Tx = y$.

## 2. BOUNDED LINEAR OPERATORS

Let $X$ and $Y$ be two vector spaces over $\mathbb{F}$. Recall that a map $T: X \to Y$is a linear operator (usually called a linear transformation in linear algebra) if for all $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{F}$,

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2).$$

The null space (or kernel) of $T, N(T)$, is the set $\{x \in X: Tx = 0\}$ and the range of $T$ is denoted by $R(T)$. Both $N(T)$ and $R(T)$ are subspaces of $X$ and $Y$ respectively.

The collection of all linear operators from $X$ to $Y$ forms a vector space $L(X, L)$under pointwise addition and scalar multiplication of functions.

When $X = \mathbb{F}^n$ and$Y = \mathbb{F}^m$, any linear operator (or called linear transformation) can be represented by an $m \times n$ matrix with entries in $\mathbb{F}$. The vector space $L(\mathbb{F}^n, \mathbb{F}^m)$ is of dimension $mn$.

When $X$ and $Y$ are normed, one prefers to study continuous linear operators. $T \in L(X,Y)$is continuous means it is continuous as a mapping from the metric space $X$ to the metric space $Y$ . It is called a bounded linear operator if it maps any bounded set in $X$ to a bounded set in. By linearity, it suffices to map a ball to a bounded set.

**Proposition (2.1)*:*** Let $T \in L(X,Y)$ where $X$ and $Y$ are normed spaces. We Have

(a) $T$ is continuous if and only if it is continuous at a point.

(b) $T$is bounded if and only if there exists a constant $C > 0$ such that

$$\|Tx\| \leq C\|x\|, \ \ for\ all\ x.$$

(c) T is continuous if and only if $T$ is bounded.

We denote the collection of all bounded linear operators from $X$ to $Y$ by

$B(X,Y)$. It is a subspace of$L(X,L)$. They coincide when $X$ and $Y$ are of finite dimension, of course.

We observe that $X' = B(X,\mathbb{F})$

The space $B(X,Y)$ not only inherits a vector space structure from $X$ and $Y$but also a norm structure. For $T \in B(X,Y)$, define its operator norm by

$$\|T\| \equiv \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

It is immediate to check that $\|\cdot\|$ makes $B(X,Y)$ into a normed space. Furthermore, for $T \in B(X,Y)$and $S \in B(Y,Z)$, the composite operator $ST \in B(X,Z)$and

$$\|ST\| \leq \|S\|\|T\|.$$

The following proposition is useful in determining the operator norm.

**Proposition (2.2)*:*** Let $T \in B(X,Y)$. Suppose $M$is a positive number satisfying

(a) $\|Tx\| \leq M\|x\|$, for all $x \in D$ where $D$ is a dense set in $X$, and

(b) there exists a nonzero sequence $\{x_k\} \subset D$ such that $\frac{\|Tx_k\|}{\|x_k\|} \to M$.Then $M = \|T\|$.

***Proof*:** For any $x \in X$, pick a sequenc $y_k \to x, y_k \in D$. Then $\|Tx\| = \lim_{k\to\infty}\|Ty_k\| \leq M \lim_{k\to\infty}\|y_k\| = M\|x\|$ shows that $\|Tx\| \leq M\|x\|$, for all $x \in X$. By the definition of the operator norm,

$$\|T\| \leq \sup_{\|x\|=1} \|Tx\| \leq M.$$

On the other hand, for the sequence $\{x_k\}$ given in (b),

$$M = \lim_{k\to\infty} \frac{\|Tx_k\|}{\|x_k\|} \leq \|T\|,$$

so $M = \|T\|$.

The following result, can be established in a similar way.

**Proposition (2.3):** $B(X,Y)$is a Banach space if $Y$ is a Banach space.Let $T \in B(X,Y)$where $X$ and $Y$ are normed spaces. Then $T$ is called invertible if it is bijective with the inverse in $B(Y,X)$In many applications, the problem can be rephrased to solving the equation $Tx = y$ in some spaces.

The invertibility of $T$ means the problem has a unique solution for every y.

Furthermore, for two solutions$Tx_i = y_i, i = 1,2$,, the continuity of $T$ implies the estimate$\|T(x_2 - x_1)\| \leq C\|y_2 - y_1\|$, from which we see that the solutions depend continuously on the given data. This is related to the so-called well- posed problem in partial differential equations.

The following general result is interesting.

**Theorem (2.4)*:*** Let $T \in B(X,Y)$ be invertible where $X$ is a Banach space.

Then $S \in B(X,Y)$ is invertible whenever $S$satisfies $\|I - T^{-1}S\|, \|I - ST^{-1}\| < 1$.

The idea is as follows. We would like to solve $Sx = y$ for a given y.

Rewriting the equation in the form $Tx + (S - T)x = y$ and applying the inverse operator to get $(I - E)x = T^{-1}y$ where $I$ is the identity operator is on $B(X,X)$and$E = T^{-1}(T - S)$ is small in operator norm. So the solution x should be given by $(\sum_{j=0}^{\infty} E^j)T^{-1}y$ as suggested by the formula

$$(1 - x)^{-1} = \sum_j x^j \text{ for} |x| < 1.$$

Our proof involves infinite series in$B(X,X)$. As parallel to what is done in elementary analysis, an infinite series$\sum_k x_k, x_k \in (X, \|\cdot\|)$, is convergent if its partial sums $s_n = \sum_1^n x_k$form a convergent sequence in $(X, \|\cdot\|)$. Wenote the following criterion for convergence.

**Proposition (2.5):** An infinite series $\sum_k x_k$ in the Banach space $X$ is convergent if there exist $a_k \geq 0$ such that $\|x_k\| \leq a_k$for all $k$ and $\sum_k a_k$ is convergent.

**Proof :** We have

$$\|s_n - s_m\| = \left\|\sum_{m+1}^{n} x_k\right\| \leq \sum_{m+1}^{n} \|x_k\| \leq \sum_{m+1}^{n} a_k,$$

and the result follows from the convergence of $\sum_k a_k$ and the completeness of $X$

In particular, the series is convergent if there exists some $\rho \in (0,1)$ such that$\|x_k\| \leq \rho^k$ for all $k$.

***Corollary*:** Let $L \in B(X,X)$where $X$ is a Banach space with$\|L\| < 1$*)2.6(*

Then $I - L$ is invertible with inverse given by

$$(I - L)^{-1} = \sum_{k=0}^{\infty} L^k$$

***Proof:*** By assumption, there exists some $\rho \in (0,1)$ such that $\|L\| \leq \rho$ From

$\|L^k\| \leq \|L\|^k \leq \rho^k$and Proposition (2.5) that $\sum_{k=0}^{\infty} L^k$ converges in $B(X,X)$.

Moreover,

$$(I - L)\sum_{k=0}^{\infty} L^k = \sum_{k=0}^{\infty} (I - L)L^k = \lim_{n\to\infty} \sum_{k=0}^{\infty} (I - L)L^k = \lim_{n\to\infty} (I - L^{n+1}) = I.$$

Similarly,$\sum_{k=0}^{\infty} L^k(I - L) = I$.

**Proof of Theorem (2.6):** We adopt the notations in the above paragraph. As $\|E\| < 1$ by assumption, Corollary (2.6) implies that $\sum_{j=0}^{\infty} E^j$ is the inverse of $I - E$. Letting $x = (\sum_{k=0}^{\infty} E^j)T^{-1}y$, then $(I - E)x = T^{-1}y$; that is, $Sx = y$.

We have shown that $S$ is onto. Also it is bounded. On the other hand, from

$\|(S - T)x\| = \|(ST^{-1} - I)Tx\| \leq \|ST^{-1} - I\|\|Tx\|$we have

$$\begin{aligned}\|Sx\| &\geq |\|Tx\| - \|(S - T)x\|| \\ &\geq (1 - \|(I - ST^{-1})\|)\|Tx\| \\ &\geq \frac{(1 - \|(I - ST^{-1})\|)}{\|T^{-1}\|}\|x\|,\end{aligned}$$

So S has a bounded inverse. We have completed the proof of this theorem.

As an application let us show that all invertible linear operators form an open set in $B(X,Y)$ when $X$ is complete. Let $T_0$ be invertible. Then for each $T$ satisfying$\|T-T_0\| < \rho \equiv 1/\|T_0^{-1}\|$; we have $\|I-T_0^{-1}T\| \le \|T_0^{-1}\|\|T_0-T\| < 1$, so by this theorem $T$ is invertible. That means the ball $B_\rho(T_0)$ is contained in the set of all invertible linear operators, and consequently it is open. For an $n\times n$-matrix, its corresponding linear transformation is invertible if and only if it is nonsingular. Again a matrix is nonsingular if and only if its determinant is non-zero. As the determinant is a continuous function on matrices (as the space $\mathbb{F}^{n^2}$ ), for all matrices close to a nonsingular matrix their determinants are non-zero, so all nonsingular matrices form an open set in the vector space of all $n\times n$-matrices. Theorem (2.4) shows that this result holds in general.

For a bounded linear operator $T$ from the normed space $X$ to another normed space $Y$ there associates with a linear operator $T'$ from $Y'$ to $X'$ called the transpose of $T$. Indeed, we define $T'$ by

$$T'y'(x) \equiv y'(Tx), \quad \text{for all} y' \in Y', \qquad x \in X.$$

It is straightforward to prove the following result.

**Proposition (2.7):** Let $T'$ be de_ned as above. Then

(a) $T'$isa bounded linear operator from $Y'$ 0 to $X'$. Furthermore. $\|T'\| = \|T\|$.

(b) The correspondence $T \to T'$ is linear from $B(X,Y)$to $B(Y',X')$.

(c) If $S \in B(Y,Z)$ where $Z$ is a normed space, then $(ST)' = T'S'$.

We examine the finite dimensional situation. Let $T$ be a linear operator from $\mathbb{F}^n$ to$\mathbb{F}^m$. Let $\{e_j\}$ and $\{f_j\}$ be the canonical bases of $\mathbb{F}^n$ *and* $\mathbb{F}^m$respec- tively. We have $Tx = \sum \ a_{k_j}\alpha_j f_k$ where$x = \sum_j \ \alpha_j e_j$ , so $T$ is represented by the matrix $m\times n$-matri$(a_{k_j})$. On the other hand, we represent $T'$ as a matrix with respect to the dual canonical bases $\{f_j'\}$ and $\{e_j'\}$ as $T'y' = \sum \ b_{k_j}\beta_j e_k'$ where$y' = \sum_j \ \beta_j f_j'$ . From the relation $T'y'(e_j) = y'(Te_j)$ for all j we have $b_{k_j} = a_{k_j}$. Thus the matrix of $T'$ is the transpose of the matrix of $T$. This justifies the terminology. In some books it is called the adjoint of $T$. Here we shall reserve this terminology for a later occasion.

There are close relations between the ranges and kernels of $T$ and those of its transpose which now we explore. Recall that the kernel of $T \in B(X,Y)$ is given by $N(T) = \{x \in X: Tx = 0\}$ and its range is $R(T) = T(X)$. The null space is always a closed subspace of $X$ and $R(T)$ is a subspace of , but it may not be closed.

For a subspace $Y$ of the normed space $X$, we define its annihilator to be

$$Y^\perp = \{x' \in X': x'(y) = 0, \quad \text{for all } y \in Y\}.$$

Similarly, for a subspace $G$of $X'$, its annihilator is given by

$$^\perp G = \{x \in X: x'(x) = 0, \qquad \text{for all} x' \in G\}.$$

It is clear that the annihilators in both cases are closed subspaces, and the following inclusions hold:

$$Y \subset^\perp (Y^\perp),$$

and

$$G \subset (\ ^\perp G)^\perp.$$

**Lemma (2.8)***:* Let $X$ be a normed space, $Y$ a closed subspace of $X$ and $G$ a closed subspace of $X'$. Then

(a)

$$Y =^\perp (Y^\perp);$$

(b) in addition, if $X$ is reflexive,

$$G \subset (\ ^\perp G)^\perp.$$

Proof. (a) It suffices to show $\ ^\perp(Y^\perp) \subset Y$. Any $x_0 \in^\perp (Y^\perp)$ satisfies $\Lambda x_0 = 0$

Whenever $\Lambda$ vanishes on$Y$ . By the spanning criterion (or Theorem 3.9), $x_0$ belongs to$Y$ .

(b) It suffices to show$({}^{\perp}G)^{\perp} \subset G$. Any $\Lambda_1 \in ({}^{\perp}G)^{\perp}$ satisfies $\Lambda_1 x = 0$ for all $x \in^{\perp} G$. If$\Lambda_1$does not belong to $G$, as $G$ is closed and the space is reexive, there is some $x_1 \in X$ such that $\Lambda_1 x_1 \neq 0$and $x_1 \in^{\perp} G$.

**Proposition (2.9)*:*** Let $X$ and $Y$ be two normed spaces and$T \in B(X, Y)$. Then we have

$$N(T') = \overline{R(T)}^{\perp},$$

$$N(T) = \overline{R(T')},$$

$$^{\perp}N(T') = \overline{R(T)},$$

$$N(T)^{\perp} = (^{\perp}\overline{R(T')})^{\perp}.$$

***Proof:*** $T'y_0' = 0$means$T'y_0'(x)$ for all $x \in X$. By the definition of the transpose of $T$ we have $y_0'(Tx) = 0$ for all $x$. Since $T$ is continuous, $y_0' \in \overline{R(T)}^{\perp}$.

We conclude that $N(T') \subset \overline{R(T)}^{\perp}$ By reversing this reasoning we obtain the other inclusion, so the first identity holds.

The second identity can be proved in a similar manner.

The third and the fourth identities are derived from the first and the second after using the previous lemma.

It is clear that we have

**Corollary (2.10)*:*** *Let $X$ and $Y$ be normed and $T \in B(X, Y)$. Then $R(T)$ is dense in $Y$ if and only if $T'$ is injective.*

The significance of this result is evident. It shows that in order to prove the solvability of the equation $Tx = y$ for any given$y \in Y$ , it suffices to show that the only solution to $T'y' = 0$ is $y' = 0$. This sets up a relation between the solvability of the equation $Tx = y$ and the uniqueness of the transposed equation $T'x = 0$.

## 3. AN APPLICATION TO FOURIER SERIES

Many details not given here.

If $f: [0,1] \to \mathbb{C}$, then the Fourier series of $f$ is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int} \qquad (3.1)$$

where the Fourier coefficients are given by

$$\hat{f}(n) = \int_{[0.1]} f(t) e^{-2\pi int} dt \qquad (3.2)$$

There are details missing from this. One is the restrictions on$f$. Using the

Lebesgue integral we would naturally impose the restriction that $f$is integrable, that is that $f$ is measurable and

$$\int_{[0.1]} |f(t)| dt < \infty$$

Apart then from a discussion of almost everywhere equivalence classes of functions, that means that $f \in L^1[0,1]$ is the natural place to consider Fourier series.

(If we restrict more to $f \in L^2[0,1]$ we get to the situation where we are considering an orthonormal basis $\{\phi_n\}^{n\in\mathbb{Z}}$of the Hilbert space $L^2[0,1]$ – with

$$\phi_n(t) = e^{2\pi int}$$

Then the Fourier series is an example of the expansion of a vector in a Hilbert space with respect to an orthonormal basis because

$$\hat{f}(n) = \langle f, \phi_n \rangle.$$

In this setting the general theory of orthonormal bases applies and quite a few strong consequences follow. However, one can argue that the $L^1[0,1]$ setting is the natural generality to use.)

Results that can be proved include:

(i) If $f, g \in L^1[0,1]$ have the same Fourier series (meaning that if the coefficients agree, $\hat{f}(n) = \hat{g}(n)$for all $n \in \mathbb{Z}$) then $f = g$ in $L^1[0,1]$(and recall that means that $f$ and $g$ agree almost everywhere).

(ii) (Riemann-Lebesgue Lemma) For $f \in L^1[0,1]$,

$$\lim_{|n|\to\infty} \hat{f}(n) = 0$$

(iii) Combining these two facts with some elementary ones, we can state it as that the map

$$T: L^1[0,1] \to c_0$$

given by

$$Tf = \left(\hat{f}(0), \hat{f}(1), \hat{f}(-1), \hat{f}(2), \hat{f}(-2), \dots\right) \qquad (3.3)$$

is a well-defined injective bounded linear operator.

Notice that the first statement says that the Fourier series of a function $f \in L^1[0,1]$ determines the function, though it does not say how to reconstruct the function.

The Riemann-Lebesgue Lemma needs some background information to prove it, principally that finite trigonometric polynomials, which take the form

$$\sum_{n=-N}^{N} a_n e^{2\pi int}$$

for $0 \le N < \infty$ are dense in $L^1[0,1]$. It is easy to compute that the Fourier coefficients $\hat{p}(n)$ for such a trigonometric polynomial $p(t) = \sum_{n=-N}^{N} a_n e^{2\pi int}$

are $\hat{p}(n) = a_n$ for $-N \le n \le N$ and $\hat{p}(n) = 0$ for other $n$. In particular note that

$Tp \in c_0$ always.

It is also easy to see (using the triangle inequality) that

$$\left|\hat{f}(n)\right| \le \|f\|_1 = \int_{[0,1]} |f(t)|dt \qquad (3.4)$$

and so $T$ maps $L^1[0,1]$ into $\ell^\infty$. It is easy to check that $T: L^1[0,1] \to \ell^\infty$is linear ($T(f+g) = Tf + Tg$ and $T(\lambda f) = \lambda Tf$ for $f, g \in L^1[0,1]$and$\lambda \in \mathbb{C}$).

Thus (3.4) gives that $T: L^1[0,1] \to \ell^\infty$ is a bounded linear operator mapping the dense subset of trigonometric polynomials into$c_0$, hence having range in $c_0$.

A natural question is to ask for a characterization of those series that have the right form to be Fourier series which are in fact the Fourier series of $L^1[0,1]$functions. The first guess might be that the Riemann-Lebesgue Lemma tells the whole story, that$T(L^1[0,1]) = c_0$.

**Theorem (3.1)*:*** *The range of the operator $T: L^1[0,1]$defined by (3.3) is a proper subspace of$c_0$.*

While it is possible to establish this result in several ways, a relatively painless way is to use the open mapping theorem. Since $T$ is injective, if it was surjectivethen its inverse would be bounded. However, there is a 'relatively' simple way to contradict that possibility:

**Lemma (3.2)*:*** *Let$D_N(t) = \sum_{n=-N}^{N} e^{2\pi int}$. Then $\|T(D_N)\|_\infty = 1$ but*

$$\lim_{N\to\infty} \|D_N\|_1 = \infty.$$

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